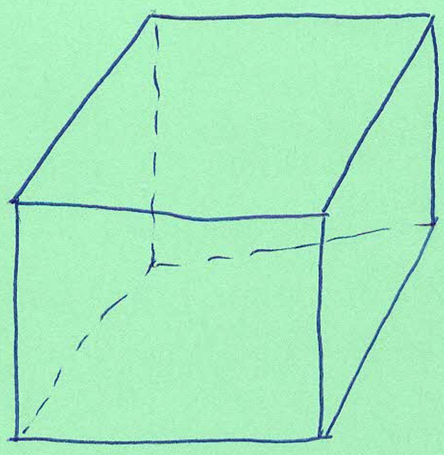
Let's do a few examples

Ex: Find the Euler Characteristic of a cube.

Sol:



vertices = 8

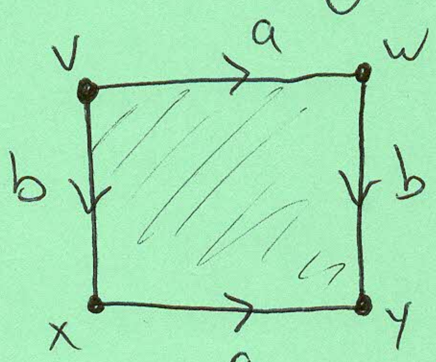
edges = 12

faces = 6

$$\chi(\text{cube}) = 6 - 12 + 8 = 2$$

Ex: Find the Euler characteristic of a torus.

Sol: The torus is given by



vertices = 1

edges = 2

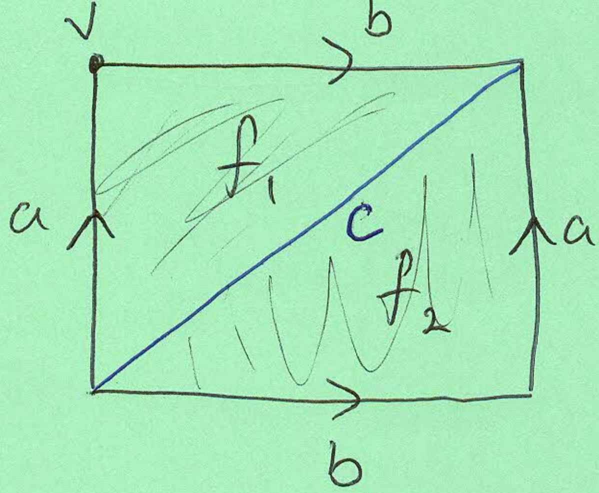
faces = 1

There is one face (the inside of the square). There are only two edges since they get identified in pairs (a & b). To check the # of vertices, start at one of them and chase it: $v \underset{\text{tail of } b}{=} w \underset{\text{tip of } a}{=} y \underset{\text{tip of } b}{=} x$. So, there is one vertex.

So, the Euler characteristic is

$$\chi(\mathbb{T}) = 1-2+1 = 0.$$

What changes if we add an edge?



faces = 2
edges = 3
vertices = 1

The edge c divide the original face into two new ones. The vertices have not changed. So, doing the Euler characteristic again:

$$\chi(\mathbb{T}) = 2-3+1 = 0.$$

Still zero!

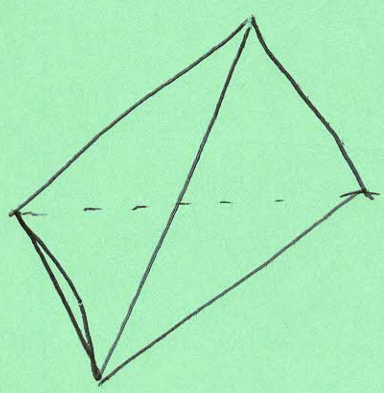
This is a manifestation of the following theorem.

<u>Theorem</u>: For a surface Σ, the Euler characteristic $\chi(\Sigma)$ does not depend on the pattern of polygons drawn on Σ.

Earlier, we saw $\chi(\text{cube}) = 2$.

<u>Ex</u>: Find the Euler characteristic of the tetrahedron.

<u>Sol</u>:



$V = 4$

$E = 6$

$F = 4$

$$\chi(\text{tetrahedron}) = 4 - 6 + 4 = 2$$

<u>Ex</u>: Find $\chi(S^2)$ (the sphere).

<u>Sol</u>: There's a whole variety of ways to do this one: soccer ball pattern $(32 - 90 + 60 = 2)$
(12 pent, 20 hex)
football pattern $(4 - 4 + 2 = 2)$
basketball pattern $(8 - 12 + 6 = 2)$

All give that $\chi(S^2) = 2$.

Notice that all of these "spherelike" or hole-less objects have Euler characteristic = 2.

Euler's Formula

Any convex polyhedron satisfies the formula

$$V - E + F = 2$$

(A polyhedron is convex if any two points in it can be connected with a line that stays inside the polyhedron.)

[This condition "convex polyhedron" can be relaxed to "surface homeomorphic to a sphere".]

<u>proof</u>: We will prove this alongside an example for clarity. Let's call our polyhedron P.

Pf:

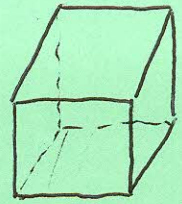
① Remove 1 face from the polyhedron.

② Flatten the rest into the plane by stretching the boundary of the removed face.

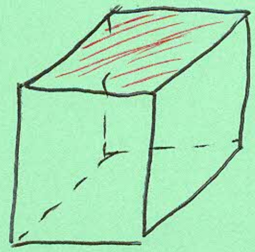
③ Add edges to divide all faces into triangles (remember, this doesn't affect the Euler characteristic)

To do this, choose two vertices of a face which are not connected by an edge and add an edge between them.

Ex: Cube



①



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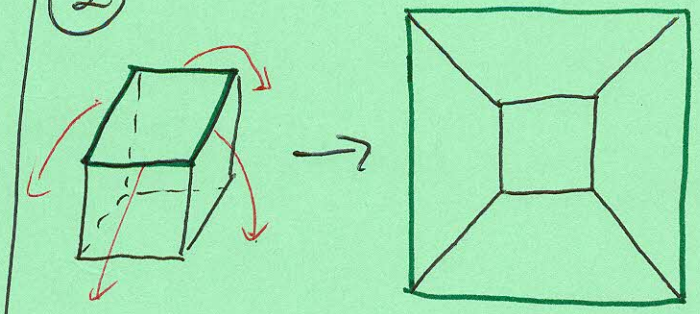

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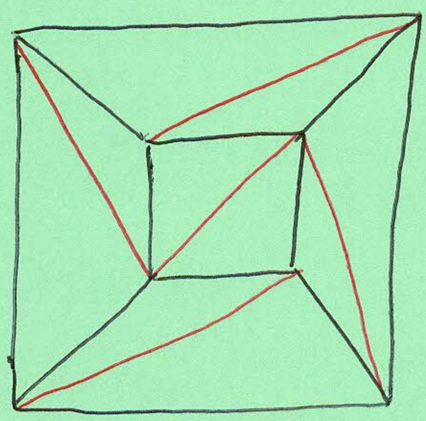
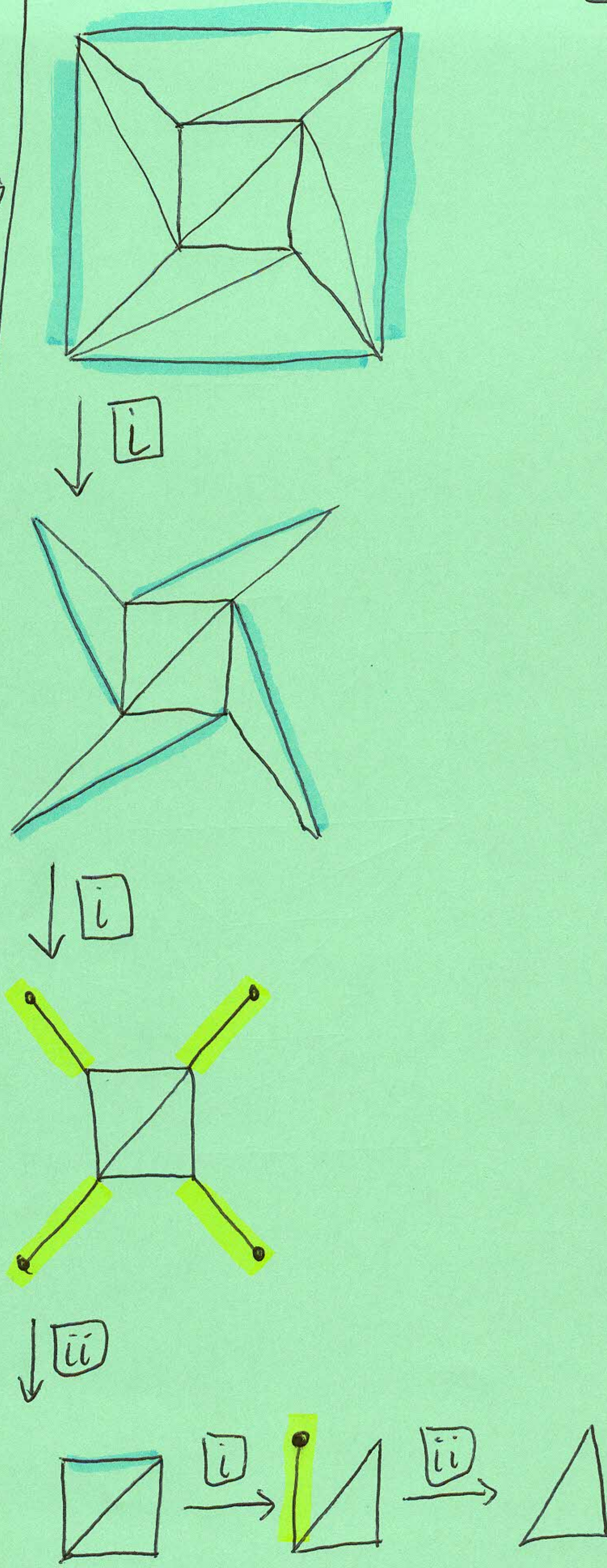


(4) Perform the following two operations as necessary to leave only a triangle remaining (or, you can go all the way to a point, if you like):

[i] Remove an exterior edge and the face it is an edge of. [Since we remove one edge & one face, the number V−E+F does not change.]

[ii] Remove an edge not bounding a triangle and the loose vertex at the end of it. [Again, since we're removing one edge and one vertex, the number V−F+F does not change.]



Once we're at a triangle we have

$$\chi(\Delta) = V - E + F = 3 - 3 + 1 = 1$$

But, remember, we deleted a face at the beginning, so

$$\chi(P) = \chi(\Delta) + \overset{\text{the deleted face}}{\downarrow \atop 1} = 2$$

$\square$

We say that a polyhedron is <u>regular</u> if all faces are congruent regular polygons. A polyhedron is a <u>Platonic solid</u> if it is convex and regular.

<u>Theorem</u>: There are exactly 5 Platonic solids.

<u>proof</u>: Let P be a Platonic solid. Then, Euler's polyhedron formula tells us

$$\chi(P) = V - E + F = 2$$

Suppose that the faces of P have n sides and that c faces meet at every vertex.